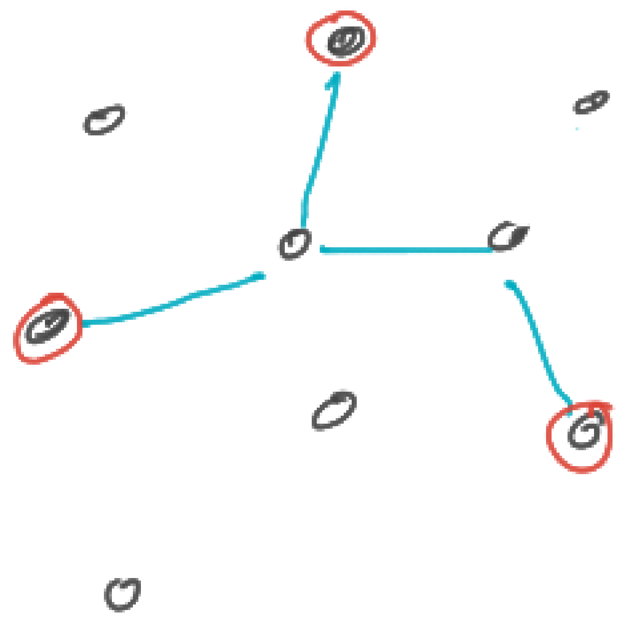
Network Design Problems

Given a graph G with edge costs and capacities
And given a collection of Demands b/w vertex pairs
Find the minimum cost subgraph that satisfies all the demands without violating any capacities.

Very General Problem, many variants
- what kind of costs and capacities?
- vertices or edges?
- directed or undirected?
- what kind of demands? Flow or distance or ??
- multicommodity or not?

Steiner Tree

Given an undirected graph $G(V,E)$, Terminals $S \subseteq V$, edge costs $c(e)\ \forall e \in E$. Find min-cost tree T that connects all terminals.



- non-terminal vertices are called Steiner vertices
- A simple greedy 2-approximation
- APX-hard ($\exists$ constant δ st no approx factor smaller than δ unless $P=NP$)
  - $95/96$ best known lower bound.
- We use ideas from Metrics once again.

Metric Completion of a graph

Given a graph G with edge costs c_e, it's metric completion is the complete graph H on $V(G)$ with edge costs

$c_{(u,v)}$ = cost of shortest path in G ($c_e \to$ length of e)

Min Spanning Tree Approx Algo

- $(H, c) \leftarrow$ Metric Completion of (G, c)
- $T \leftarrow$ Min Spanning Tree of $(H[S], c)$
- output T

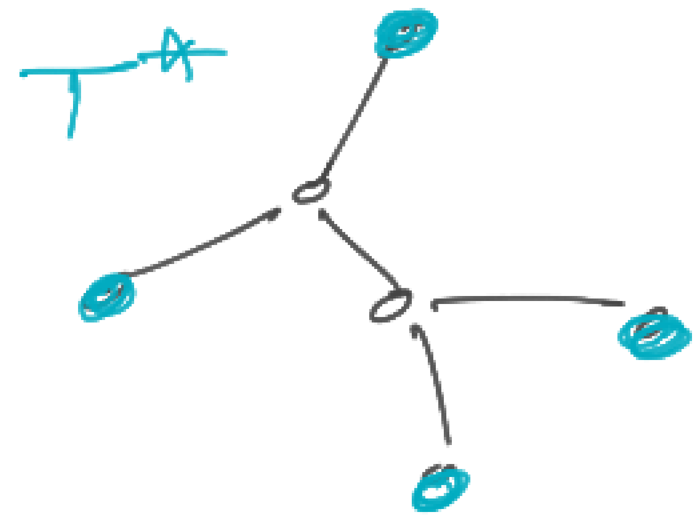
Thm: The algorithm outputs a tree of cost 2 OPT

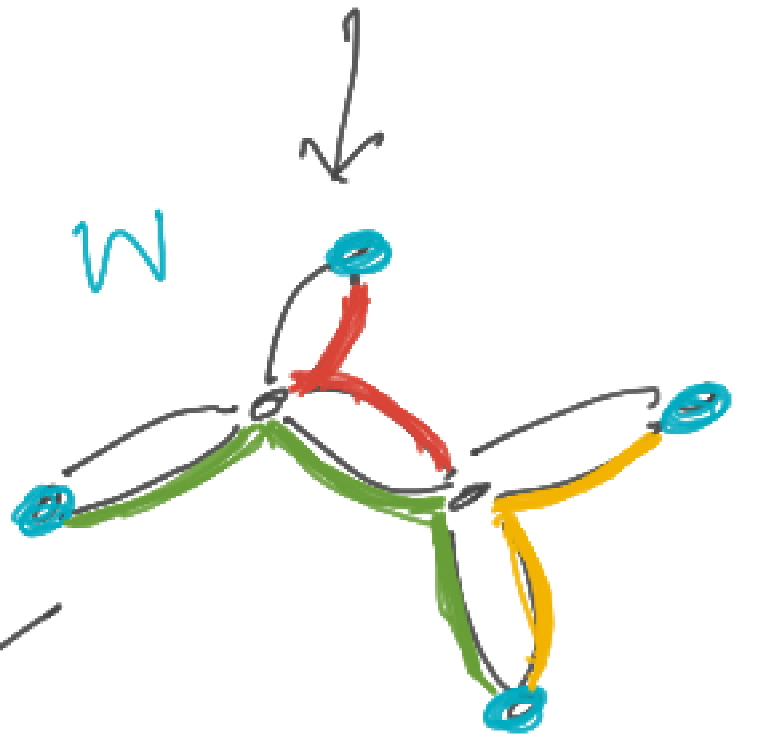
Proof: $T^* \to$ optimal Steiner Tree

$W \to$ Eulerian walk by doubling every edge of T^*

$W' \to$ metric complete edges of W to visit each vertex once

$W'' \to$ another metric completion to eliminate non-terminals.





$$c(W'') \leq c(W') \leq 2c(T^*)$$

Undirected Edge Connected Survivable Network Design

Given an undirected graph $G(V,E)$ with edge costs c_e and edge-connectivity requirements $r(u,v)$ $\forall$ $u,v \in V$

Find subgraph H minimizing $c(H)$ such that $\forall$ u,v, the edge-connectivity in H is at least $r(u,v)$

— Can you reduce Steiner Tree to this problem?

- $f : 2^V \to \mathbb{Z}$ where $f(S) = \max_{u \in S, v \in \bar{S}} r(u,v)$
- Goal: Find min-cost H st $\underbrace{\delta_H(S) \geq f(S)}$

H is an f-cover

- f is Weakly-Supermodular if

$f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$

OR

$f(A) + f(B) \leq f(A-B) + f(B-A)$

LP: $\min \sum_{e \in E} c_e x_e$

$x(\delta(S))$

st $\forall S \subseteq V \quad \sum_{e \in \delta(S)} x_e \geq f(S)$

and $x_e \in [0,1] \;\; \forall e$

without these constraint we can take multiple copies of an edge giving fake solutions

Key Thm: Let x be a basic feasible sol^n to this LP. Then $\exists\, e$ such that either $x_e = 0$ or $x_e \geq 1/2$

Algo: - $A = \phi$, $g = f$

- while A is not feasible, x = BFS to g-cover in $G - A$
  - if $\exists\, e$ st $x_e = 0$ $\quad G \leftarrow G - e$
  - else $\exists\, e$ st $x_e \geq 1/2$ $\quad A \leftarrow A \cup \{e\}$
  - $g \leftarrow f_A$ where $f_A(S) = f(S) - |\delta_A(S)|$
- Output A

Thm: The algo outputs a 2-approx sol^n

Proof Sketch: Induction on $|A|$ at output, holds when $|A| = 1$

Suppose it holds for $|A| = k$

Now consider $|A| = k+1$, $G' = G - e_1$, $f' = f_{e_1}$, $A' = A - e_1$

we have $c(A') \leq 2 \sum_{e \in G'} c_e x'_e \leq 2 OPT_{G'}$

$x' = LP\text{-}OPT$ in G', f'

x is feasible for G', f'

$\Rightarrow \sum_{e \in G'} c_e x'_e \leq \sum_{e \in G'} c_e x_e$

$\therefore c(A) = c(A') + c(e_1) \leq 2 \sum_{e \in G'} c_e x_e + 2 c_{e_1} x_{e_1}$

$\leq 2 OPT$

Key Th^m: Let x be a BFS to the LP (where f is weakly supermodular)
Then there is an $e \in E$ such that $x_e = 0$ or $x_e \geq 1/2$

- Given a sol^n x to the LP, $S \subseteq V$ is tight wrt x if

$$x(\delta(S)) = f(S)$$

Th^m: For any BFS x to the LP, $\exists\ \mathbb{L} \subseteq 2^V$ such that

- every $S \in \mathbb{L}$ is tight wrt x
- The collection of characteristic vectors $\{\chi_{\delta(S)} \mid S \in \mathbb{L}\}$ is linearly independent

  ↳ $|E|$ length 0/1 vectors

- $\mathbb{L}$ is a laminar family and $|\mathbb{L}| = |E|$

  → The inclusion relation is a forest

$\forall A, B \in \mathbb{L}$
either $A \cap B = \emptyset$
or $A \subseteq B$ or $B \subseteq A$

Proof of Key Thm:

- Suppose not, and hence $\forall e \in E$, $0 < x_e < 1/2$
- We design a Charging Scheme for $S \in \mathbb{L}$ to get a contradiction
  - $\forall e = (u,v)$ give charge $(1 - 2x_e)$ to the least $S \in \mathbb{L}$ that contain both u and v
  - $\forall e = (u,v)$ give charge x_e to least $S \in \mathbb{L}$ containing u. Likewise for v
  - Max charge given for $e \leq 1 - 2x_e + x_e + x_e = 1$

    $\Rightarrow$ Total charge given $\leq |E|$
  - However, every $S \in \mathbb{L}$ is tight $\Rightarrow \delta(S) \neq \phi$ for a largest $S \in \mathbb{L}$

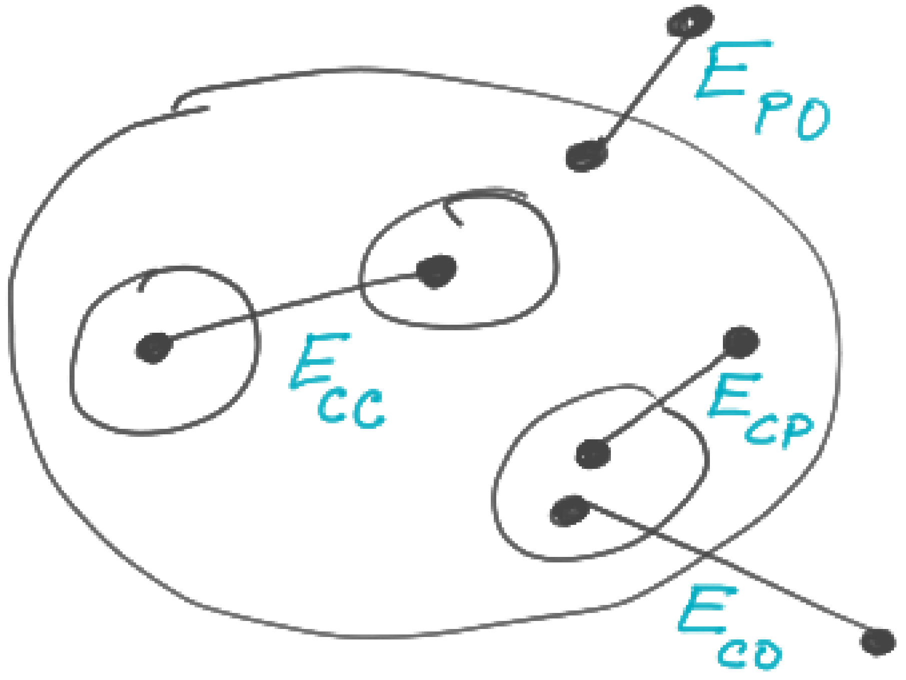
    $\therefore \exists\, e = (u,v)$ such that one endpoint of e not in any $S \in \mathbb{L}$

    $\Rightarrow$ Total charge $< |E|$

- Next, we argue that each $S \in \mathbb{L}$ gets charge ≥ 1

  As $|\mathbb{L}| = |E|$, total charge $\geq |E|$ which is a contradiction

- $\mathbb{L}$ is a laminar family $\Rightarrow$ $\mathbb{L}$ has forest like structure
- Consider $S \in \mathbb{L}$ with children $C_1 \cdots C_k$
- $E_S \rightarrow$ edges incident on S
- Partition E_S into 4 types:



- <u>Claim:</u> $E_S \setminus E_{CO} \neq \phi$

  <u>Pf:</u> Otherwise $e \in \delta(S) \Rightarrow e \in \delta(C_i)$

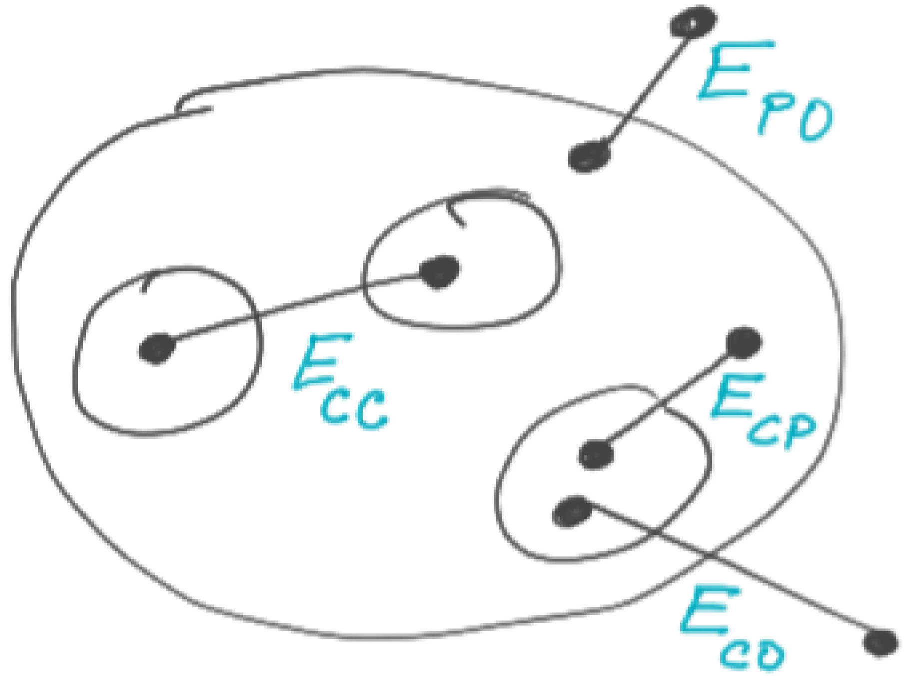
  $$\Rightarrow \quad x_{\delta(S)} = \sum x_{\delta(C_i)}$$

  violates the Linear Independence Property

- $\therefore E_{CC} \cup E_{CP} \cup E_{PO} \neq \phi \Rightarrow$ charge $(S) > 0$

- Total charge of S

$$= |E_{cc}| - 2x(E_{cc}) + |E_{cp}| - \cancel{2}x(E_{cp}) + \cancel{x(E_{cp})}$$
$$+ x(E_{po})$$



$$= |E_{cc}| + |E_{cp}| + \left(x(\delta(S)) - \sum x(\delta(C_i))\right)$$
$$= |E_{cc}| + |E_{cp}| + \left(f(S) - \sum f(C_i)\right)$$

All terms above are integral

$\therefore$ charge (S) is an integer > 0

$\Rightarrow$ charge $(S) \geqslant 1$ as required

Thm: For any BFS x to the LP where f is weakly supermodular and x is fully fractional
$\exists\ \mathbb{L} \subseteq 2^V$ such that:

- every $S \in \mathbb{L}$ is tight wrt x
- $\{\chi_{\delta(S)} \mid S \in \mathbb{L}\}$ is linearly independent
- $\mathbb{L}$ is laminar and $|\mathbb{L}| = |E|$

Uncrossing Lemma

Lemma: Given two tight sets $A, B \in 2^V$, one of the following holds:

- $A \cup B$ and $A \cap B$ are tight and $\chi_{\delta(A)} + \chi_{\delta(B)} = \chi_{\delta(A \cup B)} + \chi_{\delta(A \cap B)}$
- $A-B$ or $B-A$ are tight and $\chi_{\delta(A)} + \chi_{\delta(B)} = \chi_{\delta(A-B)} + \chi_{\delta(B-A)}$

f is weakly supermodular $\Rightarrow$ $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$

OR

$f(A) + f(B) \leq f(A-B) + f(B-A)$

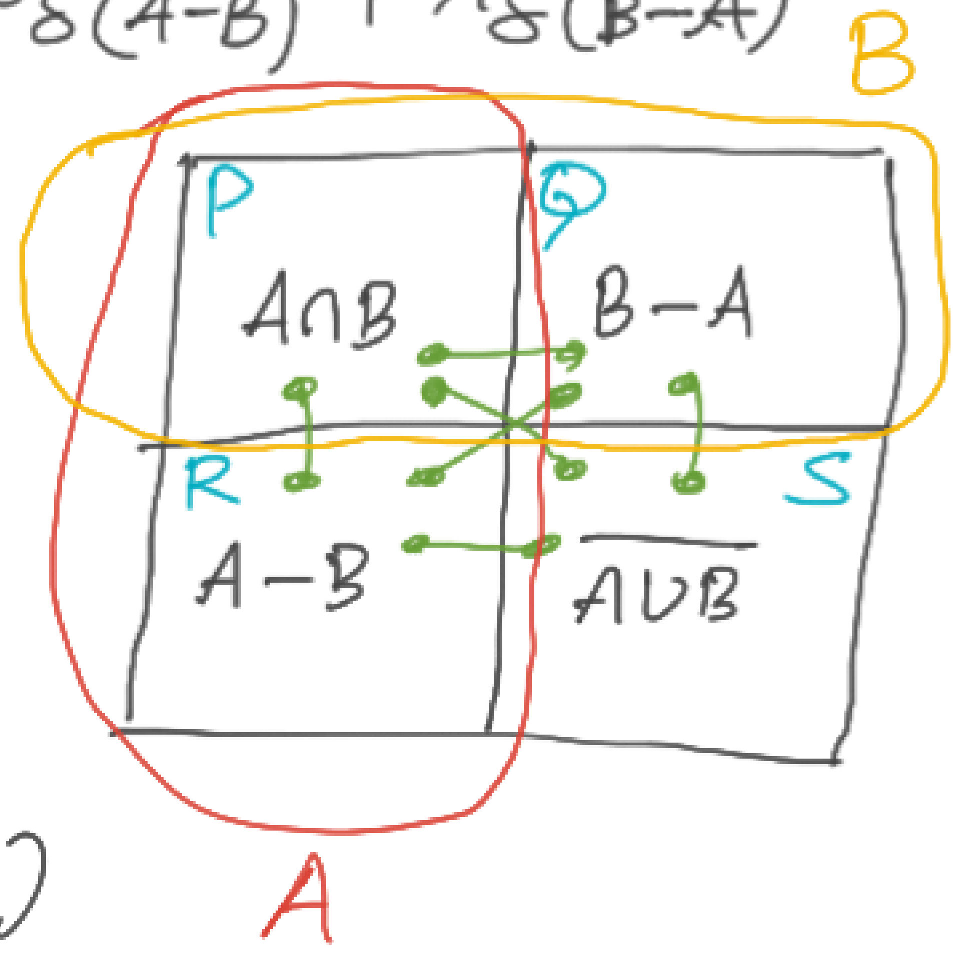
<u>Lemma:</u> Given two tight sets $A, B \in 2^V$, one of the following holds:

- $A \cup B$ and $A \cap B$ are tight and $\chi_{\delta(A)} + \chi_{\delta(B)} = \chi_{\delta(A\cup B)} + \chi_{\delta(A\cap B)}$
- $A-B$ or $B-A$ are tight and $\chi_{\delta(A)} + \chi_{\delta(B)} = \chi_{\delta(A-B)} + \chi_{\delta(B-A)}$

<u>Proof:</u> A, B tight $\Rightarrow f(A) = x(\delta(A))$ $f(B) = x(\delta(B))$

$$x(A) = x(\overline{PQ} + \overline{PS} + \overline{RS} + \overline{RQ})$$
$$x(B) = x(\overline{PR} + \overline{PS} + \overline{RQ} + \overline{QS})$$

if $f(A) + f(B) \leq f(A\cup B) + f(A \cap B)$ → other case is similar



$$\Rightarrow x(A) + x(B) \leq f(A\cup B) + f(A\cap B)$$
$$\Rightarrow x(A\cup B) + x(A\cap B) + 2x(RQ) \leq f(A\cup B) + f(A\cap B)$$

But LP feasibility $\Rightarrow$ $x(A\cup B) \geq f(A\cup B)$, $x(A\cap B) \geq f(A \cap B)$

$\therefore$ $x(A\cup B) = f(A\cup B)$ $\quad x(A\cap B) = f(A\cap B)$

$x(RQ) = 0 \Rightarrow \overline{RQ} = \emptyset \Rightarrow \chi_A + \chi_B = \chi_{A\cup B} + \chi_{A\cap B}$

Proof of Th^m:

- $\mathbb{T} \longrightarrow$ all tight sets wrt BFS x of the LP
- $\mathbb{L} \rightarrow$ a maximal collection of tight sets such that:
    - $\mathbb{L}$ is laminar
    - $\{\chi_{\delta(S)} \mid S \in \mathbb{L}\}$ is linearly independent

- Claim: $\mathrm{Span}(\mathbb{L}) = \mathrm{Span}(\mathbb{T})$

  Pf: If not $\exists\ T \in \mathbb{T}$ st $\chi_{\delta(T)} \notin \mathrm{Span}(\mathbb{L})$
  - As $\mathbb{L}$ is maximal and $\chi_{\mathbb{L}} \cup \{\chi_{\delta(T)}\}$ is lin ind

    we have $\mathbb{L} \cup \{T\}$ is not laminar
  - $\exists\ S \in \mathbb{L}$ st $S - T$, $T - S$, $S \cap T$, $\overline{S \cup T}$ are all non-empty
    - Using the Uncrossing Lemma, suppose, $\longrightarrow$ similar arguments in the other case.
    - $S \cup T$, $S \cap T$ are both tight
    - $\chi_S + \chi_T = \chi_{S \cap T} + \chi_{S \cup T}$

S and T CROSS

- As $\chi_T \notin \text{Span}(\mathbb{L})$, $\Rightarrow$ one of $\chi_{S \cup T}$, $\chi_{S \cap T} \notin \text{span}(\mathbb{L})$
- Say it is $S \cup T$ $\longrightarrow$ other case is similar
  - $S \cup T$ doesn't cross $S \in \mathbb{L}$
  - since $\mathbb{L}$ is laminar, if $S' \in \mathbb{L}$ crosses $S \cup T$ it also crosses T
- If we choose $T \in \Pi$ that crosses the fewest sets in $\mathbb{L}$, we arrive at a contradiction, since $S \cup T$ also fits the bill.
- Hence $\text{span}(\mathbb{L}) = \text{span}(\Pi)$
- For $\mathbb{L}$ we have the following properties:
  - every $S \in \mathbb{L}$ is tight wrt x
  - $\chi_{\mathbb{L}}$ is linearly independent
  - $\mathbb{L}$ is a laminar family
- We assume that $\forall e \in E$ $0 < x_e < 1$
  otherwise, if $x_e = 0$ or $x_e = 1$ our Key-Thm is already true — we can ignore this case.

- Since $x \in [0,1]^{|E|}$ is a fully fractional BFS

  By Rank Lemma, there are $|E|$ tight constraints

  $\Rightarrow$ span $(\mathbb{T})$ has dim $|E|$

  $\Rightarrow$ span $(\mathbb{L})$ has dim $|E|$

  $\Rightarrow$ $|\mathbb{L}| = |E|$

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